Adams operators and knot decorations.

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Abstract

We use an explicit isomorphism from the representation ring of the quantum group $U_q(sl(N))$ to the Homfly skein of the annulus, to determine an element of the skein which is the image of $\psi_m(c_1)$, the mth Adams operator on the fundamental representation. This element is a linear combination of m very simple m-string braids.

Using this skein element, we show that the Vassiliev invariant of degree n in the power series expansion of the $U_q(sl(N))$ quantum invariant of a knot coloured by $\psi_m(c_1)$ is the canonical Vassiliev invariant with weight system $W_n \circ \psi_m^{(n)}$, where W_n is the weight system for the Vassiliev invariant of degree n in the expansion of the quantum invariant of the knot coloured by c_1 and $\psi_m^{(n)}$ is the Adams operator on n-chord diagrams defined by Bar-Natan [4].

1 Introduction.

A motivation for this article and [1, 3] was to investigate the extent to which the properties of the Homfly polynomial and the $U_q(sl(N))$-quantum invariants are determined by the combinatoric properties of Young diagrams. This extends the work of Morton [17]. The central role of Young diagrams in both the study of symmetric polynomials and the representation theory of Lie algebras is described in many texts, for example [16, 8].

Quantum groups, introduced by Drinfel'd [5], are 1-parameter deformations of classical Lie algebras. We are concerned in particular with the quantum group $U_q(sl(N))$. For generic q, Lusztig [15] showed that the representation theory of $U_q(sl(N))$ is isomorphic to that of the classical Lie algebra. Thus the Young diagrams play a role in the world of quantum groups.

The Homfly polynomial [7, 21] is an oriented link invariant which can be described combinatorially in terms of skein relations. We will work with a framed 3-variable version, $\mathcal{X}(x, v, s)$. At certain evaluations of the variables, Turaev [24]

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showed $\mathcal{X}$ is the $U_q(sl(N))$-invariant of a knot coloured by the fundamental representation. Further, there are patterns in the Homfly skein of the annulus for which the Homfly polynomial of the satellite of a knot, at these values, is the quantum invariant of the knot coloured by some, higher dimensional, representation. Since Young diagrams index these representations, they play an integral role in Homfly skein theory.

The layout of this paper is as follows. Section 2 recalls some facts about Young diagrams. Section 3 discusses Homfly skein theory. In Sect. 4 we recall the relevant results of [3, 2]. Section 5 constructs the isomorphism between the representation ring of the quantum groups and the Homfly skein of the annulus. In Sect. 6 the mth Adams operator of the fundamental representation, $\psi_m(c_1)$ is presented. Our main result, Theorem 6.8, gives the image of $\psi_m(c_1)$ under the isomorphism of Sect. 5 as a linear combination of m very simple m-string braids. In Sect. 7 we prove that the Adams operators give rise to canonical Vassiliev invariants whose weight systems are defined using the chord diagram Adams operators of Bar-Natan[4].

Throughout, $\Lambda = \mathbb{Q}[x^{\pm 1}, v^{\pm 1}, s^{\pm 1}]$ will be the ring of Laurent polynomials in x, v and s and we will denote $(s^i - s^{-i})/(s - s^{-1})$ by $[i]$.

2 The ring of Young diagrams.

There is a wealth of detail about the features of Young diagrams in many texts (for example [8, 16]). Many of the properties to the fore in this paper are discussed in [3] and we will not repeat them here. However, we use an example to fix notation.

Let ν be the Young diagram $(4, 2, 1)$. The transpose of ν is $\nu^\vee = (3, 2, 1, 1)$. The Young tableau $T(\nu)$ is the assignment of numbers to the cells of ν shown Fig. 1 and the permutation π_ν is defined as $\pi_\nu(i) = j$ where transposition takes the cell i in $T(\nu)$ to the cell j in $T(\nu^\vee)$. Thus $\pi_\nu = (2\,4\,7\,3\,6\,5)$.

$$T(\nu) \;=\; \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & \multicolumn{2}{c}{} \\ \cline{1-2} 7 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$$

Figure 1: The tableau $T(\nu)$.

The set of formal Λ-linear combinations of Young diagrams can be given an associative, commutative algebra structure. The identity is the partition (0) and the structure constants are taken to be the Littlewood-Richardson coefficients $\{a^\nu_{\lambda\mu}\}$. These can be defined as follows. Let $\lambda = (\lambda_1, \cdots, \lambda_k)$ and $\mu = (\mu_1, , \cdots, \mu_m)$ be two Young diagrams. A *μ-expansion* of λ is obtained by first adding μ_1 boxes to λ, each labelled with a 1. No two boxes can be placed in the same column and the result must be a legitimate Young diagram. Then add μ_2 boxes labelled 2 (respecting the same rules) and continue until you have added μ_m boxes labelled

m. At each stage no two cells with the same label can appear in the same column. For any given cell, let n_i be the number of cells numbered i above and to the right of it (including the cell itself). The expansion is called *strict* if, for every cell, $n_i \geq n_j$ whenever $i < j$. The coefficient $a^{\nu}_{\lambda\mu}$ is the number of ways ν can be obtained from λ by a strict μ-expansion. This coefficient is also the multiplicity of the irreducible $U_q(sl(N))$-representation indexed by ν in the decomposition of the tensor product of the representations indexed by λ and μ, however, our definition is independent of N.

It is well known (see for example [16]) that the algebra of Young diagrams Y is freely generated as a polynomial algebra by the Young diagrams with a single column. If we denote the Λ-algebra of polynomials in an infinite number of indeterminates, $\{c_i\}_{i\in\mathbb{N}}$, by $\mathcal{R}_\infty$ then, since they are both freely generated on a countably infinite set of generators, $\mathcal{R}_\infty$ is isomorphic to Y, via the algebra isomorphism

$$\phi \ : \ c_i \quad \longmapsto \quad \left.\begin{array}{|c|}\hline \\ \hline \\ \hline \vdots \\ \hline \\ \hline \end{array}\right\updownarrow i \, .$$

Let $\mathcal{R}_N$ denote the representation ring of the classical complex Lie algebra $sl(N)$ (and thus by Lusztig also the representation ring of the quantum group $U_q(sl(N))$ for generic q). The elements of the ring are indexed by the elements of Y and the following relations hold:

i) A representation indexed by a Young diagram with a column containing more than N cells is $0 \in \mathcal{R}_N$.

ii) A representation indexed by a Young diagram with a column containing exactly N cells is isomorphic to the representation indexed by the diagram with that column removed.

Thus, we have an algebra homomorphism, $p_N : \mathcal{R}_\infty \to \mathcal{R}_N$, taking c_k to the kth exterior power of the fundamental representation. Any element of $\mathcal{R}_\infty$, which is sent, by ϕ to a single Young diagram, is mapped under p_N to an irreducible representation. The set of Young diagrams with fewer than N rows is in one-to–one correspondence with the irreducible representations. Thus, $\mathcal{R}_N$ is isomorphic to $\mathcal{R}_\infty / \langle c_k \ \forall k > N \quad c_N - 1\rangle$ and generated by $\{c_1 \cdots c_{N-1}\}$.

Let d_l denote the element of $\mathcal{R}_\infty$ whose image under ϕ is (l). Then $p_N(d_l)$ is the lth symmetric power of c_1. In particular $d_1 = c_1$ and $d_0 = c_0 = 1$.

Consider the strict expansions of a Young diagram with a single column by one with a single row. If we add the cells from the single row to the single column each cell must be placed in a different column. Hence, there are only two possibilities; one cell is added to the first column and all the others start new columns or all the cells from the single row start a new column. The corresponding diagrams have the shape of a hook. Since ϕ is an isomorphism, we shall use c_k and d_l to denote both the elements of $\mathcal{R}_\infty$ and their associated Young diagrams. For

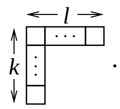


Figure 2: The Young diagram $\mu_{k,l}$.

$k, l > 0$, write $\mu_{k,l} \in Y$ for the Young diagram in Fig. 2, with $k + l - 1$ cells. Then,

$$c_k d_l = \begin{cases} c_k & \text{if } l = 0 \\ d_l & \text{if } k = 0 \\ \mu_{k+1,l} + \mu_{k,l+1} & \text{otherwise.} \end{cases} \tag{1}$$

Note that $\mu_{k,1} = c_k$ and that $\mu_{1,l} = d_l$. The following classical result plays an integral role in what follows.

2.1 Proposition.[25]

Let $C(X) = \displaystyle\sum_{k=0}^{\infty}(-1)^k c_k X^k$ and $D(X) = \displaystyle\sum_{l=0}^{\infty} d_l X^l$ be formal power series with coefficients in Y. These series satisfy the relation $C(X)D(X) = 1$.

Proof. Let $C(X)D(X) = \sum_{m=0}^{\infty} a_m X^m$ with $a_m = \sum_{k=0}^{m}(-1)^k c_k d_{m-k}$. Then $a_0 = c_0 d_0 = 1$. For $m > 0$,

$$\begin{aligned} a_m &= \sum_{k=0}^{m}(-1)^k c_k d_{m-k} \\ &= d_m + \sum_{k=1}^{m-1}\big((-1)^k(\mu_{k+1,m-k} + \mu_{k,m-k+1})\big) \quad + \quad (-1)^m c_m \\ &= d_m + \sum_{k=2}^{m}(-1)^{k-1}\mu_{k,m-k+1} \quad + \quad \sum_{k=1}^{m-1}(-1)^k \mu_{k,m-k+1} \quad + \quad (-1)^m c_m \\ &= d_m + (-1)^{m-1}\mu_{m,1} + (-1)\mu_{1,m} + (-1)^m c_m \\ &= d_m - d_m + (-1)^{m-1}(c_m - c_m) \\ &= 0. \end{aligned}$$

∎

Let $C_N(X)$ denote the polynomial obtained from $C(X)$ by taking coefficients to lie in $\mathcal{R}_N$. Thus, $C_N(X) = 1 - c_1 X + \cdots + (-1)^N c_N X^N$. This polynomial can be formally factorised as $C_N(X) = \displaystyle\prod_{i=1}^{N}(1 - x_i X)$ and each $c_k \in \mathcal{R}_N$ can be written,

formally, as the kth elementary symmetric function in $\{x_i\}_{i=1}^N$. For example, $c_1 = \sum_{i=1}^{N} x_i$. The algebra $\mathcal{R}_N$ can, therefore, be presented as the quotient of the algebra of symmetric polynomials in N variables by the ideal generated by the $c_N - 1$. However, for the purposes of this paper if we wish to work with a particular degree of polynomial, we can choose N large enough so that we do not need to be concerned with this extra relation. Under this interpretation, d_l becomes the lth complete symmetric polynomial (the sum of all the monomials of degree l). Both the set of elementary symmetric polynomials and the complete symmetric polynomials generate the algebra of symmetric polynomials. In Sect. 4 we shall give skein theoretic version of both these generating sets. We then go on to give skein theoretic version of a third generating set, the power sums.

3 The Homfly skein.

We give a brief description of skein theory based on planar pieces of knot-diagrams and a framed version of the Homfly polynomial. The ideas go back to Conway and have been substantially developed by Lickorish and others. A fuller version of this account can be found in [17, 3].

We shall work with the framed Homfly polynomial $\mathcal{X}$. This is an invariant of framed oriented links, constructed from the Homfly polynomial by setting $\mathcal{X}(L) = (xv^{-1})^{\omega(D)}P(L)$, where $\omega(D)$ is the writhe of any diagram D of the framed link L which realises the chosen framing by means of the 'blackboard parallel'. With the normalisation that $\mathcal{X}$ takes the value 1 on the empty knot, $\mathcal{X}$ is uniquely determined by the skein relations in Fig. 3.

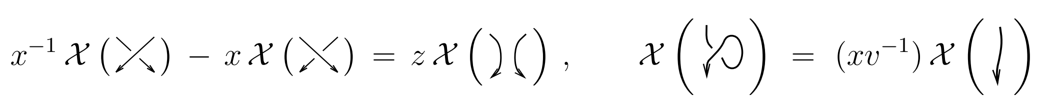


Figure 3: The framed Homfly skein relations.

Let F be a planar surface and consider diagrams in F consisting of oriented arcs joining any distinguished boundary points and oriented closed curves, up to regular isotopy. They carry the implicit framing defined by the parallel curves in the diagram. Define the *framed Homfly skein* of F, denoted by $\mathcal{S}(F)$, to consist of linear combinations of diagrams in F modulo the skein relations in Fig. 3. A detailed description of the general theory appears in [17] but here we are interested in three specific cases.

When F is the whole plane $\mathbb{R}^2$ then $\mathcal{S}(\mathbb{R}^2)$ is just the set of linear combinations of framed link diagrams, modulo the skein relations. Every diagram D represents a scalar multiple, namely $\mathcal{X}(D)$, of the empty diagram.

Let $R_n^n \cong I \times I$ be the rectangle with n distinguished points on its top and bottom edge. We insist that any arcs in R_n^n enter at the top and leave at the bottom. Diagrams in R_n^n are termed *oriented n-tangles*, and include the case of n-string braids.

Originally defined by Elrifai and Morton [6], for each permutation $\pi \in S_n$ there is an n-string *positive permutation braid* (which we shall abbreviate to p.p.b.), ω_π, uniquely determined by the fact that it is the minimal length braid, with all crossings positive, which joins the point i at the top to $\pi(i)$ at the bottom. The *negative permutation braid*(n.p.b.) $\overline{\omega}_\pi$ is defined in exactly the same manner, except that all the crossing are negative. Morton and Traczyk [20] proved that the the $n!$ n-string p.p.b.s are a linear basis for $\mathcal{S}(R_n^n)$.

The skein forms an algebra over Λ with multiplication derived from the concatenation of diagrams: write ST for the diagram obtained by placing S above T. The resulting algebra is a quotient of the braid-group algebra, shown in [20] to be isomorphic to the Hecke algebra H_n of type A, with the explicit presentation

$$H_n \quad = \quad \left\langle \quad \sigma_i \ : \ i = 1, \ldots, n-1 \qquad \left| \quad \begin{array}{l} \sigma_i\sigma_j = \sigma_j\sigma_i \ : \ |i-j| > 1 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \\ x^{-1}\sigma_i - x\sigma_i^{-1} = z \ , \end{array} \right. \right\rangle ,$$

where σ_i denotes the p.p.b. with permutation $(i\ i+1)$.

A *wiring* W of a surface F into another surface F' is a choice of inclusion of F into F' and a choice of a fixed diagram of curves and arcs in $F' - F$ whose boundary is the union of the distinguished sets of F and F'. A wiring W determines naturally a Λ-linear map $\mathcal{S}(W) : \mathcal{S}(F) \to \mathcal{S}(F')$.

We can wire the rectangle R_n^n into the annulus as indicated in Fig. 4, this being our third surface. The resulting diagram in the annulus is called the *closure* of

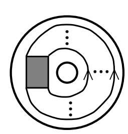
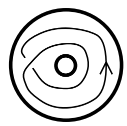
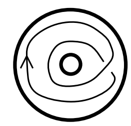


Figure 4: The wiring of $\mathcal{S}(R_n^n)$ into $\mathcal{S}(S^1 \times I)$

the oriented tangle. We shall also use the term ‘closure’ for the family of Λ-linear maps from the Hecke algebras H_n to the skein of the annulus induced by this wiring.

The skein of the annulus, $\mathcal{S}(S^1 \times I)$, itself forms an algebra, the product given by stacking the annuli one inside the other. This is obviously commutative (lift the inner annulus up and stretch it so that the outer one will fit inside it). Write $\mathcal{C}$ for $\mathcal{S}(S^1 \times I)$ regarded as a Λ-algebra in this way. Turaev [23] showed that $\mathcal{C}$ is freely generated as an algebra by $\{A_m, m \in \mathbb{Z}\}$, where A_m is the closure of the

p.p.b. for the cycle $(1\,2\,\ldots\,|m|)$ and depending on whether $m > 0$ or $m < 0$, we take the closure to be oriented in the same or the opposite sense to the core of the annulus. (For an alternative account see Hoste and Kidwell [10].) For example

$A_2 =$  and $A_{-2} =$  .

The identity element of the algebra, A_0, is represented by the empty diagram. The sub-algebra $\mathcal{C}^+$, spanned by the closures of oriented tangles using the wiring

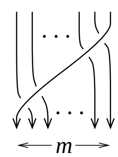


Figure 5: The braid which closes to $A_{\pm m} \in \mathcal{C}$.

shown in Fig. 4, is freely generated by $\{A_m : m \geq 0\}$. Set $\mathcal{C}^{(n)}$ to be the linear space generated by all closures of n-string oriented tangles (i.e. the image of $\mathcal{S}(R_n^n)$ under the wiring of Fig. 4). The set of monomials $(A_{i_1})^{j_1}(A_{i_2})^{j_2}\cdots(A_{i_p})^{j_p}$ where $i_k, j_k \in \mathbb{N}$ and $\sum_{k=1}^{p} i_k j_k = n$ forms a linear basis for $\mathcal{C}^{(n)}$. (Note that the number of such monomials is equal to the number of partitions of n.) Thus, the algebra $\mathcal{C}^+$ is graded by weighted degree, $\mathcal{C}^+ = \bigoplus_{n=0}^{\infty} \mathcal{C}^{(n)}$.

4 Idempotents.

In [2, 3], we adapt the construction of Gyoja [9] (similar to that of Young symmetrisers) to construct an idempotent element of $\mathcal{S}(R_n^n)$, for each Young diagram with n cells. Our building blocks are Jones idempotents [11] corresponding to single row and column Young diagrams. Equivalent idempotent elements appear in the work of Yokota [26]. For completeness, we give the bones of the construction. For a full description, see [2, 3]. Define a_n and b_n to be

$$a_n = \sum_{\pi \in S_n} (-a)^{-l(\pi)} \omega_\pi \ , \quad b_n = \sum_{\pi \in S_n} (-b)^{-l(\pi)} \omega_\pi \ ,$$

where $a = -xs^{-1}$ and $b = xs$ are the roots of the quadratic relation in H_n and $l(\pi)$ is the length of the permutation π (which is equal to the writhe of the p.p.b. ω_π). In diagrams, we will represent the element a_l by a white box labelled l and b_k by a shaded box labelled k. A tensor product sign will denote the juxtaposition of two oriented tangles. The elements a_l and b_k have the following properties.

4.1 **Lemma.**[17]

Let ϕ_a and ϕ_b be linear homomorphisms from $\mathcal{S}(R_n^n)$ to the ring of scalars Λ defined by $\phi_a(\sigma_i) = a$ and $\phi_b(\sigma_i) = b$. Then for all $T \in \mathcal{S}(R_n^n)$,

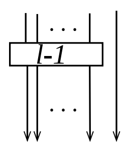
$$a_n T = T a_n = \phi_b(T) a_n\,, \quad b_n T = T b_n = \phi_a(T) b_n\,.$$

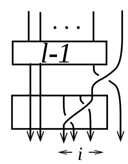
■

4.2 **Lemma.**[2]

In H_n, we can decompose a_l into a linear combination of terms involving a_{l-1}:

$$a_l \;=\; a_{l-1} \otimes a_1 + \sum_{i=0}^{l-2} (x^{-1}s)^{i+1} \;\; a_{l-1}\sigma_{l-1}\sigma_{l-2}\cdots\sigma_{l-i-1}$$

$=$ 

$+ \sum_{i=0}^{l-2} (x^{-1}s)^{i+1}$ 

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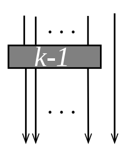
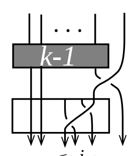
Similarly,

$$b_k \;=\; b_{k-1} \otimes b_1 + \sum_{i=0}^{k-2} (-x^{-1}s^{-1})^{i+1} \;\; b_{k-1}\sigma_{k-1}\sigma_{k-2}\cdots\sigma_{k-i-1}$$

$=$ 

$+ \sum_{i=0}^{k-2} (-x^{-1}s^{-1})^{i+1}$ 

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Equally, we can decompose a_l and b_k from above (turn the pictures upside down).

■

We now define a quasi-idempotent element e_λ for each $\lambda \in Y$. Assign a braid string to each cell, ordered by the tableau $T(\lambda)$. On the strings in the ith row of λ, we place a_{λ_i}. We will denote this linear combination of braids by $E_\lambda(a)$. Similarly define $E_\lambda(b)$, replacing a_{λ_i} by b_{λ_i}. For example, if $\nu = (4,2,1)$ then $E_\nu(a) = a_4 \otimes a_2 \otimes a_1$. Define $e_\lambda = E_\lambda(a)\omega_{\pi_\lambda} E_{\lambda^\vee}(b)\omega_{\pi_\lambda}^{-1}$, where π_λ is the permutation defined in Sect. 2. The element e_ν is shown in Fig. 6.

4.3 **Theorem.**[2, 1]

The e_λ, $|\lambda| = n$, are quasi-idempotent, mutually orthogonal elements of $\mathcal{S}(R_n^n)$ i.e. $e_\lambda^2 = \alpha_\lambda\, e_\lambda$ for some scalar α_λ and for $\lambda \neq \mu$, $e_\lambda e_\mu = 0$. We will denote the

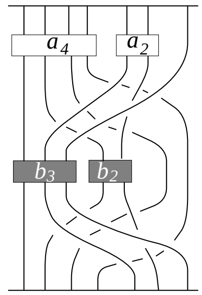

Figure 6: The quasi-idempotent e_ν.

closures of the genuine idempotents, $1/\alpha_\lambda\ e_\lambda$, by $Q_\lambda \in \mathcal{C}^+$. The Q_λ form a free Λ-basis for $\mathcal{C}^+$. The value for α_λ is given by the formula

$$\alpha_\lambda = \prod_{\text{cells}} s^{\text{content}}[\text{hook length}] = \prod_{(i,j)\in\lambda} s^{j-i}[\lambda_i + \lambda_j^\vee - i - j + 1]$$

Denote the $U_q(sl(N))$ invariant of a knot K coloured by the representation V by $J(K;V)$. Let C be a framed knot and let S be the satellite $C * Q_\lambda$. Set $\mathcal{X}_N$ to be the evaluation of $\mathcal{X}$ at $x = s^{-1/N}$ and $v = s^{-N}$, where $s^2 = q$. Then,

$$J(C;V_\lambda) = \mathcal{X}_N(S)\,.$$

■

A subtle point (discussed in detail in [19]) is that we require $1/\alpha_\lambda \in \Lambda$ and so we must extend the scalar ring so that the quantum integers can be inverted. From now on, we shall assume that this is the case. As they will be used in the proceeding sections, we give the values of the scalar α for the hook shaped diagrams explicitly. For these Young diagrams, it is quite simple to calculate $\alpha_{\mu_{k,l}}$ directly using Lemma 4.2.

$$\alpha_{\mu_{k,l}} = s^{(l(l-1)-k(k-1))/2}\,[\,k+l-1\,][\,k-1\,]![\,l-1\,]!\,.$$

5 An isomorphism from $\mathcal{R}_\infty$ to $\mathcal{C}^+$.

We next describe an algebra isomorphism from $\mathcal{R}_\infty$ to $\mathcal{C}^+$. We then demonstrate a generating set for $\mathcal{C}^+$ by considering the image, under this isomorphism, of the Adams operators of c_1, which generate $\mathcal{R}_\infty$ as an algebra.

To improve notation marginally, we shall adopt $e_{k,l}$ to denote the quasi-idempotent associated to the Young diagram $\mu_{k,l}$. Similarly we will write $Q_{k,l}$ and $\alpha_{k,l}$ rather than $Q_{\mu_{k,l}}$ and $\alpha_{\mu_{k,l}}$.

5.1 Lemma.

The element $Q_{k,1}$ is an element of $\mathcal{C}^{(k)}$ and can, therefore, be expressed as a weighted homogeneous polynomial of degree k in the A_m with $k \geq m > 0$. The coefficient of A_k in $Q_{k,1}$ is $(-x^{-1})^{k-1}/[k]!$.

Sketch proof. Since $Q_{k,1}$ is the closure of a linear combination of diagrams in $\mathcal{S}(R_k^k)$, $Q_{k,1}$ is an element of $\mathcal{C}^{(k)}$, and thus weighted homogeneous of degree k, in terms of the A_m.

It can be shown by induction that $(-x^{-1})^{k-1}s^{-k(k-1)/2}[k-1]!$ is the coefficient of A_k in b_k. For the induction step, apply Lemma 4.2 to b_k to obtain the term $b_{k-1}\otimes 1$ and a sum of k-string diagrams involving b_{k-1}. The term $b_{k-1}\otimes 1$ has A_1 as a factor thus, the monomial A_k will not appear in this expression. Repeated application of Lemma 4.2 to each of the remaining terms, along with some book keeping finishes the step. Dividing by $\alpha_{k,1}$ we obtain the coefficient of A_k in $Q_{k,1}$ as $(-x^{-1})^{k-1}/[k]$.

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5.2 Proposition.

Define the algebra homomorphism $\theta : \mathcal{R}_\infty \to \mathcal{C}^+$ by $\theta(c_k) = Q_{k,1} \quad \forall k \in \mathbb{N}$. The homomorphism θ is an algebra isomorphism.

Proof. The ring $\mathcal{R}_\infty$ is defined to be the free algebra generated by c_k, $k \in \mathbb{N}$. Hence, the homomorphism θ is an isomorphism if we can show that the elements $Q_{k,1}$ freely generate $\mathcal{C}^+$ as an algebra.

Recall that $\{A_m : m \in \mathbb{N}\}$, freely generate $\mathcal{C}^+$ as an algebra. By induction on n, we will show that A_n can be expressed in terms of the $Q_{k,1}$, for $0 < k \leq n$. and thus, $\{Q_{k,1} \ : \ k \in \mathbb{N}\}$ generates $\mathcal{C}^+$ as an algebra.

For $n = 1$, we have that $Q_{1,1} = A_1$, and we are done.

For $n > 1$, assume that we have an expression for A_m in terms of the $Q_{k,1}$ for all $m < n$. It follows from the definition of $\mathcal{C}^{(n)}$ that $Q_{n,1} \in \mathcal{C}^{(n)}$. Therefore, there is an expression for $Q_{n,1}$ in terms of monomials of weighted degree n in the A_m. The monomials of weighted degree n can be indexed by the partitions of n, hence, by Lemma 5.1

$$Q_{n,1} = \frac{(-x^{-1})^{n-1}}{[n]} A_n + \sum_{|\lambda|=n} \beta_\lambda A_{\lambda_1} A_{\lambda_2} \cdots A_{\lambda_k}$$

for some scalars $\beta_\lambda \in \Lambda$. Thus, by the induction hypothesis, since the coefficient of A_n is invertible in Λ, we can write A_n in terms of the weighted degree n monomials in the $Q_{k,1}$, $k \leq n$. Thus, these monomials generate $\mathcal{C}^{(n)}$ as a Λ-module. This is equivalent to saying that the $Q_{k,1}$ generate $\mathcal{C}^+$ as an algebra.

It remains to show that $\mathcal{C}^+$ is generated freely by $Q_{k,1}$, $k \in \mathbb{N}$. We have just shown that the set of monomials $Q_{i_1,1}^{j_1} Q_{i_2,1}^{j_2} \cdots Q_{i_m,1}^{j_m}$ for which $\sum_{p=1}^m i_p j_p = n$,

generate $\mathcal{C}^{(n)}$ as a Λ-module. By definition, $\mathcal{C}^{(n)}$ is freely generated by the set of weighted degree n monomials in the A_m, which is a set with the same cardinality. Therefore, since Λ is commutative, the monomials in the $Q_{k,1}$ of weighted degree n form a free Λ-basis for $\mathcal{C}^{(n)}$. It follows that $\{Q_{k,1} \,:\, k \in \mathbb{N}\}$ generate the algebra $\mathcal{C}^+$ freely.

■

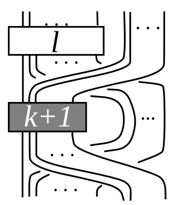
5.3 Lemma.

Let $\widehat{e}_{k,l}$ denote the closure of $e_{k,l}$ in $\mathcal{C}^+$. The following relation holds :

$$s^l[\,l\,]\widehat{e}_{k+1,l} + s^{-k}[\,k\,]\widehat{e}_{k,l+1} = s^{l-k}[\,l+k\,]\,\widehat{e}_{1,l}\,\widehat{e}_{k,1}\,.$$

Proof. For simplicity, we draw the oriented tangles which are wired into $\mathcal{C}^+$, as shown in Fig 4. However, since we are working in $\mathcal{C}^+$, we can slide a piece of diagram off the top of a diagram and reintroduce it at the bottom without loss. All strings are assumed to be oriented from top to bottom, unless indicated otherwise. Applying Lemma 4.2 to $\widehat{e}_{k+1,l}$,

$\widehat{e}_{k+1,l} \;=$



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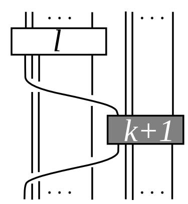

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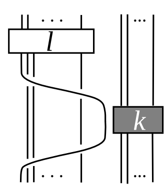
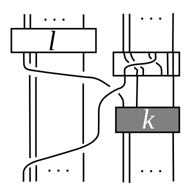


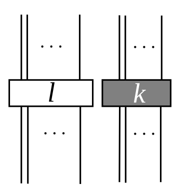
$+ \sum_{i=0}^{k-1} \Bigg((-x^{-1}s^{-1})^{i+1}$



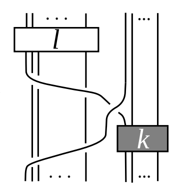
$\Bigg)\,.$

Since we are working in the skein of the annulus, by Lemma 4.1,

$\hat{e}_{k+1,l} \;=$

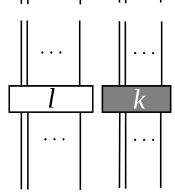


$+ \sum_{i=0}^{k-1} \Bigg((-x^{-1}s^{-1})^{i+1}(-xs^{-1})^{i}$

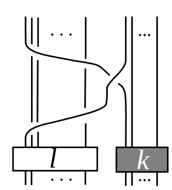


$\Bigg)$

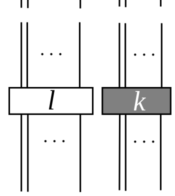
$=$



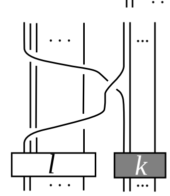
$- \;\; x^{-1}\left(\sum_{i=0}^{k-1} s^{-2i-1}\right)$



$=$

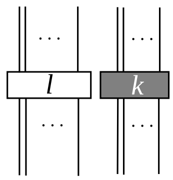


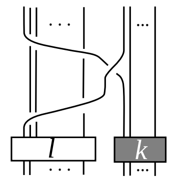
$- \;\; x^{-1}s^{-k}[\,k\,]$



$.$

Using the same techniques on $\widehat{e}_{k,l+1}$ we obtain

$\widehat{e}_{k,l+1} \quad = \quad$ 

$\quad + \quad x^{-1}s^l[\,l\,]$ 

.

Hence, eliminating the term on the far right,

$$s^l[\,l\,]\widehat{e}_{k+1,l} + s^{-k}[\,k\,]\,\widehat{e}_{k,l+1} \;=\; (s^l[\,l\,] + s^{-k}[\,k\,])\,\widehat{e}_{1,l}\,\widehat{e}_{k,1} \;=\; s^{l-k}\,[\,l+k\,]\,\widehat{e}_{1,l}\,\widehat{e}_{k,1}\,.$$

■

5.4 Proposition.

The idempotents $Q_{k,l}$ satisfy $Q_{k+1,l} + Q_{k,l+1} = Q_{k,1}Q_{1,l}$.

Proof. Substitute $\widehat{e}_{k,l} = \alpha_{k,l}\,Q_{k,l}$ into Lemma 5.3. ■

5.5 Corollary.

Let $Q_C(X)$ and $Q_D(X)$ respectively, denote the formal power series

$$Q_C(X) = \sum_{k=0}^{\infty}(-1)^k Q_{k,1}X^k \quad \text{and} \quad Q_D(X) = \sum_{l=0}^{\infty} Q_{1,l}X^l\,.$$

(These formal power series can be thought of as $\mathcal{C}^+$–versions of the formal power series $C(X)$ and $D(X)$, defined in Prop. 2.1.) Then $Q_C(X)$ is the inverse of $Q_D(X)$,

$$Q_C(X)Q_D(X) = 1\,.$$

Proof. Since Prop. 5.4 echoes Eq. (1), the proof of Corollary 5.5 is analogous to that of Prop. 2.1. ■

We next demonstrate that $\theta(\mu_{k,l}) = Q_{k,l}$. (In fact it can be shown that for any Young diagram λ, $\theta(\lambda) = Q_\lambda$, and that the Q_λ form a linear basis for $\mathcal{C}^+$, but we do not need this here. Details can be found in [2].) First, we prove the following lemma.

5.6 Lemma.

The image of the Young diagram (l) under θ is $Q_{1,l}$.

Proof. The proof goes by induction on the number of cells. We know that $d_1 = c_1$, therefore $\theta(d_1) = \theta(c_1) = Q_{1,1}$.

Assume that we have the result for d_i, for $i < m$. By Prop. 2.1, for $m \geq 1$, $\sum_{k=0}^m (-1)^k c_k d_{m-k} = 0$. Since θ is an algebra isomorphism, this implies that

$$\theta(d_m) = \sum_{k=1}^{m} (-1)^{k-1} \theta(c_k)\theta(d_{m-k}) \,. \tag{2}$$

By Prop. 5.5 and the induction hypothesis

$$\begin{aligned} 0 = \sum_{k=0}^{m} (-1)^k Q_{k,1} Q_{1,m-k} &= Q_{1,m} + \sum_{k=1}^{m} (-1)^k \theta(c_k)\theta(d_{m-k}) \\ &= Q_{1,m} - \theta(d_m) \quad \text{by Eq. (2).} \end{aligned}$$

■

5.7 Proposition.

For $k, l \geq 1$, the image of $\mu_{k,l}$ under θ, is $Q_{k,l}$.

Proof. By Lemma. 5.6, $\theta(\mu_{1,l}) = \theta(d_l)$ for $l \in \mathbb{N}$. This provides the base for an induction on the number of cells and the length of the first column. Assuming the result for all hook shaped diagrams with fewer than m cells or m cells and at most k cells in the first column,

$$\begin{aligned} \theta\left(\mu_{k+1,m-k}\right) &= \theta(c_k d_{m-k}) - \theta(\mu_{k,m-k+1}) \quad \text{by Prop. 5.2} \\ &= Q_{k,1}\, Q_{1,m-k} - Q_{k,m-k+1} \quad \text{by the induction step} \\ &= Q_{k+1,m-k} \quad \text{by Prop 5.4}\,, \end{aligned}$$

as required.

■

6 Adams operators.

Recall that we can express the elements of $\mathcal{R}_N$ as symmetric polynomials in N variables. It is well known that the power sums $\sum_i x_i^m$, $m \in \mathbb{N}$, generate the algebra of symmetric polynomials. Thus, the images of these power sums under θ, will give us an alternative generating set for $\mathcal{C}^+$. We give an expression for the mth power sum as a sum of m closed braids in $\mathcal{C}^+$. An advantage of these power sums over the $Q_{k,1}$ is that the number of braids in their expressions is linear

rather than factorial in the number of strings and all the braids are reasonably simple. Recall the definitions of C_N and p_N from Sect. 2.

The *Adams operators*, $\{\psi_m\}_{m\in\mathbb{N}}$, are a family of $\mathcal{R}_N$-endomorphisms, defined by their images on the x_i (despite the fact that the x_i are just a formal device and are not elements of $\mathcal{R}_N$),

$$\psi_m(x_i) = x_i^m.$$

Hence $\psi_m(c_1) = \sum x_i^m$ is the mth Newton power sum. As a polynomial in the c_i it is independent of N, i.e. there is a polynomial $\beta_m(\mathbf{c}) \in \mathcal{R}_\infty$ with $p_N(\beta_m) = \psi_m(c_1)$ for all N.

6.1 Proposition.

The following identities hold for $\psi_m(c_1)$,

$$\psi_m(c_1) \;=\; p_N\left(\sum_{k=1}^{m}(-1)^{k-1}kc_kd_{m-k}\right) \;=\; p_N\left(\sum_{k=1}^{m}(-1)^{k-1}\mu_{k,m-k+1}\right).$$

Proof. The function $\ln(C(X))$ has formal power series expansion

$$\ln(C(X)) = \sum_{m=1}^{\infty}\beta_m(\mathbf{c})X^m$$

where $\beta_m(\mathbf{c})$ denotes a polynomial in the c_k. Differentiating $\ln(C(X))$ with respect to X, we get

$$\frac{C'(X)}{C(X)} = C'(X)D(X) = \sum_{m=1}^{\infty} m\beta_m(\mathbf{c})X^{m-1}.$$

By comparing coefficients it follows that

$$\begin{aligned}
m\beta_m(\mathbf{c}) \;&=\; \sum_{k=1}^{m}(-1)^k kc_kd_{m-k}\\
&=\; \sum_{k=1}^{m-1}\left((-1)^k k(\mu_{k+1,m-k}+\mu_{k,m-k+1})\right) \quad + (-1)^m m\mu_{m,1}\\
&=\; (-1)^m m\mu_{m,1} + \sum_{k=1}^{m-1}(-1)^k k\mu_{k,m-k+1} + \sum_{k=2}^{m}(-1)^{k-1}(k-1)\mu_{k,m-k+1}\\
&=\; -\mu_{1,m} + (-1)^m\mu_{m,1} + \sum_{k=2}^{m-1}(-1)^k\mu_{k,m-k+1}\\
&=\; \sum_{k=1}^{m}(-1)^k\mu_{k,m-k+1} \quad .
\end{aligned}$$

For each N we have that

$$\ln(C_N(X)) \;=\; \sum_{i=1}^{N} \ln(1-x_iX) \;=\; -\sum_{m=1}^{\infty}\sum_{i=1}^{N} \frac{x_i^m}{m}X^m \;=\; -\sum_{m=1}^{\infty} \frac{\psi_m(c_1)}{m}X^m\,.$$

Now $\ln(C_N(X)) = p_N(\ln(C(X)))$, therefore, $\psi_m(c_1) = -p_N(m\beta_m(\mathbf{c}))$, as required. ■

Since these formulae are independent of N, for most purposes we can treat $\psi_m(c_1)$ as if it were an element of $\mathcal{R}_\infty$.

6.2 Lemma.

Let $\Psi(X) = \sum_{m=1}^{\infty} \psi_m(c_1)X^{m-1}$. Then

$$\Psi(X) = -C'(X)D(X) = D'(X)C(X)\,.$$

Proof. From Prop. 6.1 we know that

$$\Psi(X) = -\frac{d}{dX}\,\ln(C(X)) = -C'(X)D(X)\,.$$

It remains to note that, by Prop. 2.1

$$D'(X)C(X) = \left(\frac{1}{C(X)}\right)' C(X) = \frac{-C'(X)}{C(X)^2}C(X) = -C'(X)D(X)\,.$$

■

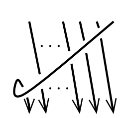
Strickland and independently Rosso and Jones [22, 12] proved the following formula for the $U_q(sl(N))$-invariants of torus knots. However, the independence of the result from the knot K implies that the relation also holds directly in $\mathcal{R}_\infty$ (and via θ in $\mathcal{C}^+$).

6.3 Theorem. [22, 12]

Let m and p be coprime integers. Let $K^{(m,p)}$ denote the (m,p) cable of the knot K. Then if $\psi_m(c_1) = \sum_{\tau\in Y} \beta_\tau \tau$ then

$$J(K^{(m,p)};c_1) = \sum_{\tau} \beta_\tau f_\tau^{p/m} J(K;\tau)\,.$$

where the (m,p)-cable of K is the satellite of K with pattern the pth power of the tangle



$\leftarrow\; m \;\rightarrow$

and f_τ is the framing factor associate to τ and $J(K;\tau)$ is the $U_q(sl(N))$ invariant of K coloured by τ. ■

By Prop. 6.1 and Theorem 6.3, setting $p=1$, we obtain the following corollary.

6.4 Corollary.

$$A_m = (xv^{-1})^{-1}\sum_{k=1}^{m}(-1)^{k-1} f_{k,m-k+1}^{\frac{1}{m}}\theta(\mu_{k,m-k+1})\,,$$

where $f_{k,m-k+1}$ is the framing factor of the Young diagram $\mu_{k,m-k+1}$. ■

Originally calculated in [18], the framing factors for $U_q(sl(N))$ calculated via skein theory [3] are given by

$$f_\lambda = x^{|\lambda|^2}v^{-|\lambda|}s^{n_\lambda}\,,$$

where $n_\lambda = \sum_i \lambda_i^2 - \sum_j(\lambda_j^\vee)^2$. For $\mu_{k,m-k+1}$, $n_{k,m-k+1} = m^2-2mk+m$. Note that, since $\mu_{k,m-k+1}$ has m cells, $f_{k,m-k+1}^{\frac{1}{m}}$ contains only integer powers of x, v and s. We can obtain a similar expression for the closure of the m-string n.p.b., $\overline{A}_m$, by replacing s, x and v by s^{-1}, x^{-1} and v^{-1} respectively in the formula for A_m. To see this substitute $p=-1$ in Theorem 6.3 and note that in $\mathcal{C}^+$ the closure of $\sigma_1\sigma_2\cdots\sigma_{m-1}$ is equivalent to the closure of $\sigma_{m-1}\sigma_{m-2}\cdots\sigma_1$.

6.5 Proposition.

The following equalities hold:

$$A_m = x^{m-1}\sum_{k=1}^{m}(-1)^{k-1}s^{m-k}[k]\theta(c_kd_{m-k})\,, \tag{3}$$

$$\overline{A}_m = x^{-(m-1)}\sum_{k=0}^{m-1}(-1)^{k}s^{k}[m-k]\theta(c_kd_{m-k})\,. \tag{4}$$

Proof. By Cor. 6.4

$$\begin{aligned}
A_m &= x^{-1}v\sum_{k=1}^{m}(-1)^{k-1}x^m v^{-1}s^{m-2k+1}\theta(\mu_{k,m-k+1})\\
&= \sum_{k=1}^{m}(-1)^{k-1}x^{m-1}s^{m-2k+1}\theta(\mu_{k,m-k+1})\\
&= x^{m-1}\sum_{k=1}^{m}(-1)^{k-1}s^{m-2k+1}\sum_{i=k}^{m}(-1)^{i-k}\theta(c_id_{m-i})\\
&= x^{m-1}\sum_{i=1}^{m}\sum_{k=1}^{i}(-1)^{i-1}s^{m-2k+1}\theta(c_id_{m-i})\\
&= x^{m-1}\sum_{i=1}^{m}(-1)^{i-1}s^{m+1}\left(\sum_{k=1}^{i}s^{-2k}\right)\theta(c_id_{m-i})
\end{aligned}$$

$$
\begin{aligned}
&= x^{m-1}\sum_{i=1}^{m}(-1)^{i-1}s^{m+1}s^{-2}\frac{(1-s^{-2i})}{(1-s^{-2})}\theta(c_i d_{m-i}) \\
&= x^{m-1}\sum_{i=1}^{m}(-1)^{i-1}s^{m-i}[i]\theta(c_i d_{m-i})\,.
\end{aligned}
$$

This establishes Eq. 3. For Eq. 4, note that $\sum_{i=0}^{m}(-1)^i\theta(c_i)\theta(d_{m-i}) = 0$ by Prop. 5.6, therefore,

$$
\begin{aligned}
A_m &= A_m + \theta(x^{m-1}[m]\sum_{i=0}^{m}(-1)^i c_i d_{m-i}) \\
&= [m]x^{m-1}\theta(c_0 d_m) + x^{m-1}\sum_{i=1}^{m}(-1)^{i-1}(s^{m-i}[i]-[m])\theta(c_i d_{m-i}) \\
&= [m]x^{m-1}\theta(c_0 d_m) + x^{m-1}\sum_{i=1}^{m}(-1)^{i}\left(\sum_{k=1}^{m-i}s^{-m-1+2k}\right)\theta(c_i d_{m-i})\,.
\end{aligned}
$$

Then

$$
\begin{aligned}
\overline{A}_m &= x^{-(m-1)}\sum_{i=0}^{m}(-1)^i\left(\sum_{k=1}^{m-i}s^{m+1-2k}\right)\theta(c_i d_{m-i}) \\
&= x^{-(m-1)}\sum_{i=0}^{m}(-1)^i s^i[m-i]\theta(c_i d_{m-i}) \\
&= x^{-(m-1)}\sum_{i=0}^{m-1}(-1)^i s^i[m-i]\theta(c_i d_{m-i}).
\end{aligned}
$$

■

We define $\Phi^+(X)$ and $\Phi^-(X)$ to be the formal power series

$$
\Phi^+(X) = \sum_{m=1}^{\infty} A_m X^{m-1} \quad \text{and} \quad \Phi^-(X) = \sum_{m=1}^{\infty} \overline{A}_m X^{m-1} \quad . \tag{5}
$$

We also define "quantum derivatives" of $C(X)$ and $D(X)$,

$$
C'_q(X) = \sum_{k=1}^{\infty}(-1)^k[k]c_k\,X^{k-1} \quad \text{and} \quad D'_q(X) = \sum_{l=1}^{\infty}[l]d_l\,X^{l-1}\,.
$$

6.6 Proposition.

The following three identities hold:

$$\begin{aligned}
\Phi^+(X) &= -\,\theta\,(\,C'_q(xX)D(xsX)\,)\ , \\
\Phi^-(X) &= \theta\left(\,C(x^{-1}sX)D'_q(x^{-1}X)\,\right)\ , \\
\Phi^-(X) &= -\,\theta\left(\,C'_q(x^{-1}X)D(x^{-1}s^{-1}X)\,\right)\ .
\end{aligned}$$

Proof. The first two identities follow automatically from Prop. 6.5. To see the third identity note that, by Prop. 2.1,

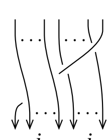
$$\sum_{m=1}^{\infty} x^{-m+1}[m]\left(\sum_{k=0}^{m}(-1)^k c_k d_{m-k}\right) X^{m-1} \quad = \quad 0\,.$$

We can, therefore, subtract any multiple of the image of this sum under θ from $\Phi^-(X)$. Hence

$$\begin{aligned}
\Phi^-(X) &= \Phi^-(X) - \sum_{m=1}^{\infty} x^{-m+1}\sum_{k=0}^{m}(-1)^k[m]\theta(c_k d_{m-k})\,X^{m-1} \\
&= \sum_{m=1}^{\infty} x^{-m+1}\sum_{k=0}^{m}(-1)^k(s^k[m-k]-[m])\theta(c_k d_{m-k})\,X^{m-1} \\
&= \sum_{m=1}^{\infty} x^{-m+1}\sum_{k=0}^{m}(-1)^k\left(\frac{s^m-s^{-m+2k}-s^m+s^{-m}}{s-s^{-1}}\right)\theta(c_k d_{m-k})\,X^{m-1} \\
&= \sum_{m=1}^{\infty} x^{-m+1}\sum_{k=0}^{m}(-1)^k s^{-m+k}\left(\frac{-s^k+s^{-k}}{s-s^{-1}}\right)\theta(c_k d_{m-k})\,X^{m-1} \\
&= \sum_{m-1}^{\infty} x^{-m+1}\sum_{k=0}^{m}(-1)^{k-1}s^{-m+k}[k]\theta(c_k d_{m-k})X^{m-1}\,.
\end{aligned}$$

The third equation now follows by comparing the coefficient of X^{m-1} above with that of X^{m-1} in $\theta\,(\,C(x^{-1}sX)D'_q(x^{-1}X)\,)$. ■

Let $A_{i,j}$ be the closure of the braid below in $\mathcal{C}^+$, with i positive crossings and j negative crossings (so $A_m = A_{m-1,0}$ and $\overline{A}_m = A_{0,m-1}$).

$A_{i,j} \;=$ 

6.7 Lemma.

Let

$$\begin{aligned} P_m &= x^{m-1} + x^{m-3} + x^{m-5} + \cdots + x^{-m+3} + x^{-m+1} \\ &= \sum_{i=0}^{m-1} x^{m-1-2i} A_{i,m-1-i} \ . \end{aligned}$$

Then $P_m = \sum_{k=1}^{m-1} (s - s^{-1}) x^{m-1-k} P_k A_{0,m-1-k} \ + m x^{m-1} A_{0,m-1}$.

Proof. The method here is to switch and smooth the positive crossings in the braids using the skein relation. Numbering the crossings from bottom left to top right on the diagrams, we start with the $(m-1)$st (positive) crossing.

$$\begin{aligned} P_m &= x^{m-1} + x^{m-3} + x^{m-5} + \cdots + x^{-m+3} + x^{-m+1} \\ &= x^{m-1} A_{0,m-1} \\ &\quad + x^{-1} \left(x^{m-2} + x^{m-4} + \cdots + x^{-m+4} + x^{-m+2} \right) \\ &= x^{m-1} A_{0,m-1} + (s - s^{-1}) P_{m-1} A_{0,0} \\ &\quad + x \left(x^{m-2} + x^{m-4} + \cdots + x^{-m+4} + x^{-m+2} \right) \\ &= 2x^{m-1} A_{0,m-1} + (s - s^{-1}) P_{m-1} A_{0,0} \\ &\quad + x^{m-3} + \cdots + x^{-m+5} + x^{-m+3} \ . \end{aligned}$$

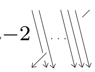
Applying the skein relation to the $(m-2)$nd crossing we then see that

$$\begin{aligned} P_m &= 2x^{m-1} A_{0,m-1} + (s - s^{-1})(P_{m-1} A_{0,0} + x P_{m-2} A_{0,1}) + x^{m-1} A_{0,m-1} \\ &\quad + \text{ weighted sum of diagrams with the } (m-1)\text{st and} \\ &\qquad (m-2)\text{nd crossings negative.} \end{aligned}$$

Applying the skein relation to all the positive $(m-3)$rd, $(m-4)$th, $\ldots$, 2nd crossings we finally arrive at

$$P_m = (m-1) x^{m-1} A_{0,m-1} + (s - s^{-1}) \sum_{k=2}^{m-1} x^{m-1-i} P_k A_{0,m-k-1} + x^{m-3}$$

$$= \quad (m-1)x^{m-1}A_{0,m-1} + (s-s^{-1})\sum_{k=2}^{m-1} x^{m-1-i}P_k A_{0,m-k-1}$$

$$+ x^{-1}\Big(x^{m-2}$$  $$\Big)$$

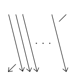
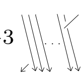
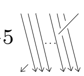
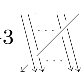
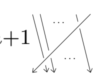
$$= \quad (m-1)x^{m-1}A_{0,m-1} + (s-s^{-1})\sum_{k=2}^{m-1} x^{m-1-k}P_k A_{0,m-k-1}$$
$$+(s-s^{-1})P_1 A_{o,m-2} + x x^{m-2} A_{o,m-1}$$

$$= \quad m x^{m-1}A_{0,m-1} + (s-s^{-1})\sum_{k=1}^{m-1} x^{m-1-k}P_k A_{0,m-k-1}$$

$\blacksquare$

6.8 Theorem.

The image of the mth Adams operator of the fundamental representation in the skein of the annulus $\mathcal{C}^+$, under θ, is a scalar multiple of

$P_m = x^{m-1}$  $+ x^{m-3}$  $+ x^{m-5}$  $+ \cdots + x^{-m+3}$  $+ x^{-m+1}$ 

namely,

$$P_m = [m]\theta(\psi_m(c_1))\,.$$

Proof. The proof goes by induction on m. For $m = 1$ we have $P_1 = c_1$ and $\psi_1(c_1) = c_1$.

Now, assuming the result holds for all $k < m$, we can apply Lemma 6.7 to obtain the following,

$$P_m = \sum_{k=1}^{m-1}(s-s^{-1})x^{m-1-k}P_k A_{0,m-1-k} \quad + m x^{m-1}A_{0,m-1}\,.$$

Therefore, by the induction hypothesis,

$$P_m = \sum_{k=1}^{m-1}(s^k-s^{-k})x^{m-1-k}\theta(\psi_k(c_1))A_{0,m-1-k} \quad + m x^{m-1}A_{0,m-1}\,.$$

Using the definition in Lemma 6.2 and Eq. (5) it can be shown that

$$\sum_{k=1}^{m-1} x^{m-1-k}(s^k-s^{-k})\theta(\psi_k(c_1))A_{0,m-k}$$

is the coefficient of X^{m-2} in

$$s\,\theta\left(\,\Psi(sX)\,\right)\Phi^-(xX) - s^{-1}\,\theta\left(\,\Psi(s^{-1}X)\,\right)\Phi^-(xX)\,.$$

Applying Lemma 6.2 and Prop. 6.6

$$\begin{aligned}
s\theta\left(\,\Psi(sX)\right)\Phi^-(xX) \;-\; & s^{-1}\theta\left(\,\Psi(s^{-1}X)\right)\Phi^-(xX)\\
=\; & s\,\theta\left(-C'(sX)D(sX)C(sx^{-1}xX){D'}_q(x^{-1}xX)\right)\\
& +s^{-1}\theta\left(D'(s^{-1}X)C(s^{-1}X)D(x^{-1}s^{-1}xX){C'}_q(x^{-1}xX)\right)\\
=\; & \theta\left(s^{-1}D'(s^{-1}X){C'}_q(X) - s{D'}_q(X)C'(sX)\right) \text{ by Prop.2.1.}
\end{aligned}$$

We will denote the preimage of the coefficient of X^{m-2} in this expression by g_{m-2}.

$$g_{m-2} = \sum_{k=1}^{m-1}(-1)^k[k](m-k)s^{k-m}c_kd_{m-k} \;-\sum_{k=1}^{m-1}(-1)^k[m-k]ks^kc_kd_{m-k}.$$

The coefficient of c_kd_{m-k} in g_{m-2} is

$$\begin{aligned}
(-1)^k\left((m-k)\frac{(s^{-m+2k}-s^{-m})}{s-s^{-1}}\right. \;-\; & \left. k\frac{(s^m-s^{-m+2k})}{s-s^{-1}}\right)\\
=\; & (-1)^k\left(ms^{-m+k}[k] - k\frac{(s^m-s^{-m})}{s-s^{-1}}\right)\\
=\; & (-1)^k(ms^{-m+k}[k]-k[m])\\
=\; & (-1)^kms^{-m+k}[k]+(-1)^{k-1}k[m]\,.
\end{aligned}$$

Recall that $A_{0,m-1} = \overline{A}_m$ and add back on the term $mx^{m-1}\theta(A_{0,m-1})$. By Prop. 6.5,

$$\begin{aligned}
P_m \;=\; & \theta\left(\sum_{k=1}^{m-1}(-1)^k(ms^{-m+k}[k]-k[m])c_kd_{m-k}\right.\\
& \left. \qquad\qquad + m\sum_{k=0}^{m-1}(-1)^ks^k[m-k]c_kd_{m-k}\right)\\
=\; & \theta\left(\sum_{k=1}^{m-1}(-1)^k\left(ms^k\frac{(s^{k-m}-s^{-k-m}+s^{m-k}-s^{k-m})}{s-s^{-1}}-k[m]\right)c_kd_{m-k}\right.\\
& \left. \qquad\qquad + m[m]c_0d_m\right)\\
=\; & \theta\left(\sum_{k=1}^{m-1}(-1)^k(m[m]-k[m])c_kd_{m-k} \quad + \quad m[m]c_0d_m\right)
\end{aligned}$$

$$
\begin{aligned}
&= \theta\left([m]\sum_{k=1}^{m}(-1)^{k-1}kc_kd_{m-k} \quad - \quad (-1)^{m-1}m[m]c_md_0 \right.\\
&\qquad\qquad\qquad\qquad \left. + m[m]c_0d_m \quad + m[m]\sum_{k=1}^{m-1}(-1)^k c_kd_{m-k}\right)\\
&= \theta\left([m]\psi_m(c_1) + m[m]\sum_{k=0}^{m}(-1)^k c_kd_{m-k}\right) \qquad \text{by Prop. 6.1}\\
&= [m]\theta\left(\psi_m(c_1)\right) + m[m]0 \qquad \text{by Prop. 2.1}\\
&= [m]\theta\left(\psi_m(c_1)\right)
\end{aligned}
$$

■

7 Vassiliev Invariants.

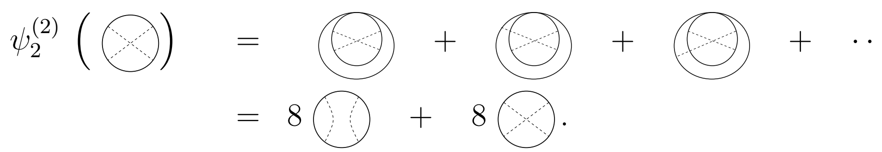
One motivation for considering the Adams operators came from the work of Bar-Natan [4] in the context of Vassiliev invariants. He defines an "Adams operator" on chord diagrams. We will denote the Adams operator defined on the subspace of n-chord diagrams by $\psi_m^{(n)}$. Here we investigate the extent to which applying our Adams operators to the colour in a quantum invariant upstairs, agrees with Bar-Natan's Adams operators on chord diagrams. For a detailed account of the theory of Vassiliev invariants the reader is referred to [4].

A fundamental result in the theory of Vassiliev invariants is that every degree n weight system comes from a type n Vassiliev invariant and this is well defined up to Vassiliev invariants of lower order. Further, there is a universal Vassiliev invariant Z satisfying $V_n(W) = W \circ Z$ for each degree n weight system W, so we can "integrate" a weight system to get a Vassiliev invariant. In fact, there is a preferred choice for Z, which we will denote $\mathbf{Z^K}$. This is the Kontsevich integral and there are several equivalent constructions for $\mathbf{Z^K}$, for example [13, 14]. Let $\mathbf{Z}^{\mathbf{K}}_n$ be the projection of $\mathbf{Z^K}$ onto the nth graded piece of the algebra of chord diagrams and suppose that V is a type n Vassiliev invariant. We say that V is *canonical* if and only if $V = W(V) \circ \mathbf{Z}^{\mathbf{K}}_n$. Let h be a formal parameter and suppose $V = \sum_{i=0}^{\infty} V_i h^i$ is a power series over h where V_i is a Vassiliev invariant of type i. We say V is *canonical* if each of the V_i is a canonical invariant.

We are interested in canonical invariants for the following reasons. Two canonical invariants are equal if and only if their weight systems are equal. Equivalently, the weight system uniquely determines the canonical invariant. The $U_q(sl(N))$-invariant $J(K; V_\lambda)$ with λ fixed is a canonical power series, when expanded at $q = e^h$ [13, 14].

Let D be an n-chord diagram. Following [4, Defn. 3.11], we define $\psi_m^{(n)}(D)$ to be the sum of all possible ways of lifting D to the mth cover of the circle. For

example,



7.1 Theorem.

Let $J(K;c_1) = \sum_{i=0}^{\infty} V_i h^i$ be the Vassiliev power series expansion of $J(K;c_1)$. Take the Vassiliev power series expansion of $J(K;\psi_m(c_1))$ to be $\sum_{i=0}^{\infty} U_i(K)h^i$. Then U_i is the canonical degree i Vassiliev invariant with weight system $W_i \circ \psi_m^{(i)}$.

$$U_i = \big(W(V_i) \circ \psi_m^{(i)}\big) \circ \mathbf{Z}_i^{\mathbf{K}}$$

Proof. By Theorem 6.8,

$$J(K;\psi_m(c_1)) \;=\; \frac{1}{[m]}\,J(K * P_m;c_1) \;=\; \frac{1}{[m]}\sum_{i=0}^{\infty} V_i(K * P_m)h^i$$

Assume that K has n double points. Then $K * P_m$ can be written as a linear combination of singular knots with n double points and $V_i(K * P_m;c_1) = 0$ when $i < n$. Further, expanding $[m]$ as a power series in h, $[m] = m + O(h)$. Thus

$$J(K;\psi_m(c_1)) \;=\; \frac{1}{[m]}\,J(K * P_m;c_1) \;=\; \frac{1}{m}\,V_n(K * P_m)h^n + O(h^{n+1}) \qquad (6)$$

Using the notation of Sect. 6, consider a summand $A_{i,m-1-i}$ of P_m. Since V_n is a Vassiliev invariant,

$$V_n(K * A_{i,m-1-i}) = V_n(K * A_{i+1,m-2-i}) \,-\, V_n(K * B_i)\,,$$

where B_i is a pattern which contains a double point. Since we assumed that K has n double points, $V_n(K * B_i) = 0$. By repeated application of this relation, we see that $V_n(K * A_{i,m-1-i}) = V_n(K * A_m)$, where $A_m (= A_{m-1,0})$ is the element depicted in Fig. 5. Since $x = 1 + O(h)$, as a power series in h,

$$V_n(K * P_m) \;=\; mV_n(K * A_m) + O(h)\,.$$

Substituting back into Eq. (6), we have

$$J(K;\psi_m(c_1)) = \frac{m}{m}V_n(K * A_m)h^n + O(h^{n+1})\,. \qquad (7)$$

Equating coefficients of h^n, Eq. 7 implies that for a knot with n double points $V_n(K * A_m) = U_n(K)$. Equivalently, for any $n \geq 0$ and any knot K, the two invariants agree at the weight system level,

$$W(V_n(K * A_m)) = W(U_n(K))\,.$$

The satellite $K * A_m$ is a connected m string cable of K. We can think of $V_n(K * A_m)$ as an invariant of K by taking the composition of V_n with the cabling map $\varphi_m : \varphi_m(K) = K * A_m$. Bar-Natan [4] showed that $W(V_n \circ \varphi_m) = W(V_n) \circ \psi_m^{(n)}$ and as we commented above, U_n is a canonical invariant, therefore, for each $n \geq 0$,

$$U_n = \left(W(V_n) \circ \psi_m^{(n)} \right) \circ \mathbf{Z}_n^{\mathbf{K}} .$$

■

We have shown that the canonical Vassiliev power series with weight system $W(J(K; c_1)) \circ \psi_m^{(n)}$ is the expansion of $J(K; \psi_m(c_1))$. (By $W(J(K; c_1))$ we mean the sum of the weight systems for each of the Vassiliev invariants in the Vassiliev power series expansion of $J(K; c_1)$.) Bar-Natan's result holds in a more general context. It says that for any type n Vassiliev invariant V,

$$W(V \circ \varphi_m) = W(V) \circ \psi_m^{(n)} .$$

In particular, if we can express $\psi_m(\lambda)$ as a linear combination of cabling patterns, coloured by λ, we could prove the following conjecture

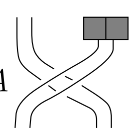
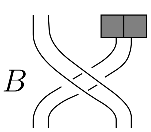
7.2 Conjecture.

Let $J(K; V_\lambda)$ have a Vassiliev power series expansion $\sum_{i=0}^{\infty} V_{i,\lambda} h^i$. Then the canonical Vassiliev invariant with weight system $W(V_{i,\lambda}) \circ \psi_m^{(n)}$ is the coefficient of h^i in the Vassiliev power series expansion of $J(K; \psi_m(\lambda))$. ■

An obvious guess is that if we colour P_m (or some other linear combination of the braids in P_m) with λ this might work. However, this doesn't work even for the simplest case, $\psi_2(c_2)$. In terms of Young diagrams,

$$\psi_2(c_2) = (4) - (2,1,1) + (2,2) .$$

We would need the following equation to hold in $\mathcal{C}^+$,

$Q_{(4)} - Q_{(2,1,1)} + Q_{(2,2)} \;=\; A$  $+\; B$  .

Expressing everything in terms of monomials in $\{A_m\}_{m \in \mathbb{N}}$, we can compare the coefficients of each of the five weighted degree four monomials in the A_m, and so solve for A and B in more than one way. Unfortunately, the answers are not consistent. Thus, there is no linear combination of connected 2-cables, P', for which $J(K; \psi_2(c_2)) = J(K * P'; c_2)$. The pattern for $\psi_2(c_2)$ must, therefore, involve the 2-parallel and make use of the Vassiliev skein relation, to relate it to the connected cablings modulo extra double points. Since $(2,2)$ is not a hook shaped diagram, perhaps this is not too surprising in light of Cor. 6.4.

References

[1] A. K. Aiston, *A skein theoretic proof of the hook formula for quantum dimension*, In preparation University of Liverpool 1996.

[2] A. K. Aiston, *Skein theoretic idempotents of Hecke algebras and quantum group invariants*, Ph.D. thesis, University of Liverpool (1996).

[3] A. K. Aiston and H. R. Morton, *Idempotents of Hecke algebras of type A*, Preprint University of Liverpool (1996) q-alg/9702017.

[4] D. Bar-Natan, *On the Vassiliev knot invariants*, Topology **34** no. 2 (1995) pp423–472.

[5] V. G. Drinfel'd, *Quantum groups*, In 'Proceedings of the I.C.M. (Berkeley 1986)', pp798–820. A.M.S. Providence R.I., 1987.

[6] E. A. Elrifai and H. R. Morton, *Algorithms for positive braids*, Quart. J. Math. Oxford **45** no. 2 (1994) pp479–497.

[7] P. Freyd, D. Yetter, J. Hoste, W. B. R. Lickorish, K. C. Millet, and A. Ocneanu, *A new polynomial invariant of knots and links*, Bull. A. M. S. **12** (1985) pp239–246.

[8] W. Fulton and J. Harris, *Representation theory – a first course*, G.T.M. Springer Verlag, 1991.

[9] A. Gyoja, *A q-analogue of Young symmetriser*, Osaka J. Math. **23** (1986) pp841–852.

[10] J. Hoste and M. E. Kidwell, *Dichromatic link invariants*, Trans. Amer. Math. Soc. **321** (1990) pp197–229.

[11] V. F. R. Jones, *Hecke algebra representations of braid groups and link polynomials*, Ann. Math. **126** (1987) pp335–388.

[12] V. F. R. Jones and M. Rosso, *On invariants of torus knots derived from quantum groups*, J. Knot Theory Ramif. **2** (1993) pp97–112.

[13] C. Kassel, *Quantum groups*, G.T.M. Springer Verlag, 1995.

[14] T. Q. T. Le and J. Murakami, *The universal Vassiliev–Kontsevich invariant for framed oriented links*, Compos. Math. **102** no. 1 (1996) pp41–64.

[15] G. Lusztig, *Quantum deformations of certain simple modules over enveloping algebras*, Adv. Math. **70** (1988) pp237–249.

[16] I. G. Macdonald, *Symmetric functions and Hall polynomials (2nd edition)*, Clarendon Press. Oxford, 1995.

[17] H. R. Morton, *Invariants of links and 3-manifolds from skein theory and from quantum groups*, In M. Bozhüyük (editor), 'Topics in knot theory. N.A.T.O. A.S.I. series C 399', pp107–156. Kluwer, 1993.

[18] H. R. Morton, *Quantum invariants given by evaluation of knot polynomials*, J. Knot Theory Ramif. **2** (1993) pp195–209.

[19] H. R. Morton and A. K. Aiston, *Young diagrams, the Homfly skein of the annulus and unitary invariants*, In S. Suzuki (editor), 'Knots 96: proceedings of the 5th International research institue of the Mathematics Society of Japan', pp31–45. World Scientific, 1997.

[20] H. R. Morton and P. Traczyk, *Knots and algebras*, In E. Martin-Peinador and A. R. Usan (editors), 'Contribuciones Matematicas en homenaje al profesor D. Antonio Plans Sanz de Bremond', pp201–220. University of Zaragoza, 1990.

[21] J. Przytycki and P. Traczyk, *Invariants of links of Conway type*, Kobe J. Math. **4** (1987) pp115–139.

[22] P. Strickland, *On the quantum enveloping algebra invariants of cables*, Preprint University of Liverpool (1990).

[23] V. G. Turaev, *The Conway and Kauffman modules of the solid torus with an appendix on the operator invariants of tangles*, LOMI preprints **E-6-88** (1988).

[24] V. G. Turaev, *The Yang Baxter equation and invariants of links*, Invent. Math. **92** (1988) pp527–553.

[25] H. Weyl, *The classical groups and their invariants and representations*, Princeton University Press, 1946.

[26] Y. Yokota, *Skeins and quantum* $SU(N)$ *invariants of* 3*-manifolds*, Math. Ann. **307** no. 1 (1997) pp109–138.